

Comparison of categorical characteristic classes of transitive Lie algebroid with Chern-Weil homomorphism

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September 3, 2012

Abstract

Transitive Lie algebroids have specific properties that allow to look at the transitive Lie algebroid as an element of the object of a homotopy functor. Roughly speaking each transitive Lie algebroids can be described as a vector bundle over the tangent bundle of the manifold which is endowed with additional structures. Therefore transitive Lie algebroids admits a construction of inverse image generated by a smooth mapping of smooth manifolds.

Due to to K.Mackenzie ([1]) the construction can be managed as a homotopy functor $\mathcal{TLA}_{\mathfrak{g}}$ from category of smooth manifolds to the transitive Lie algebroids. The functor $\mathcal{TLA}_{\mathfrak{g}}$ associates with each smooth manifold M the set $\mathcal{TLA}_{\mathfrak{g}}(M)$ of all transitive algebroids with fixed structural finite dimensional Lie algebra \mathfrak{g} . Hence one can construct ([4],[5]) a classifying space $\mathcal{B}_{\mathfrak{g}}$ such that the family of all transitive Lie algebroids with fixed Lie algebra \mathfrak{g} over the manifold M has one-to-one correspondence with the family of homotopy classes of continuous maps $[M, \mathcal{B}_{\mathfrak{g}}]$: $\mathcal{TLA}_{\mathfrak{g}}(M) \approx [M, \mathcal{B}_{\mathfrak{g}}]$.

It allows to describe characteristic classes of transitive Lie algebroids from the point of view a natural transformation of functors similar to the classical abstract characteristic classes for vector bundles and to compare them with that derived from the Chern-Weil homomorphism by J.Kubarski([3]). As a matter of fact we show that the Chern-Weil homomorphism does not cover all characteristic classes from categorical point of view.

1 Basic definitions and functor $\mathcal{TLA}_{\mathfrak{g}}(\bullet)$

1.1 Definitions

Definition 1.1.1. (See [1], Definition 3.1.1) A Lie algebroid A over a smooth manifold M is a vector bundle $p : A \rightarrow M$ together with a Lie algebra structure $\{\bullet\}$ on the space of $\Gamma^\infty(A; M)$ and a bundle map $a : A \rightarrow TM$ called the anchor, such that

- (i) the induced map $a : \Gamma(A; M) \rightarrow \Gamma(TM; M) = \mathfrak{X}^1(M)$ is a Lie algebra homomorphism
- (ii) for any sections $\alpha, \beta \in \Gamma(A; M)$ and smooth function $f \in C^\infty(M)$ we have the Leibniz identity

$$\{\alpha, f \cdot \beta\} = f \cdot \{\alpha, \beta\} + a(\alpha)(f) \cdot \beta$$

We call A a regular Lie algebroid if the rank of a is locally constant and A a transitive Lie algebroid if a is surjective. The Lie algebroid homomorphism and isomorphism is defined in [1]. And we often use the Atiyah exact sequence $0 \rightarrow L \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0$ to denote a transitive Lie algebroid. Here $L = \mathbf{Ker} a$ is called the adjoint bundle. Sometimes we use $(A, M, \{\bullet\}, a)$ to note Lie algebroid in order to highlight the bracket. All transitive Lie algebroids (isomorphic class) and homomorphisms between them form a category that is fundamental in our considerations.

Example 1.1.2. (See [1]) The followings are important examples of transitive Lie algebroid.

1. Let M be a manifold and let \mathfrak{g} be a Lie algebra. On $TM \oplus (M \times \mathfrak{g})$ define $a : TM \oplus (M \times \mathfrak{g}) \rightarrow TM$ by $a : (X, \mu) \mapsto X$. And a bracket

$$\{(X, \mu), (Y, \nu)\} = ([X, Y], X(\nu) - Y(\mu) + [\mu, \nu]).$$

for $(X, \mu), (Y, \nu) \in \Gamma(TM \oplus (M \times \mathfrak{g}); M)$.

Then $TM \oplus (M \times \mathfrak{g})$ is a transitive Lie algebroid on M , called the trivial Lie algebroid on M with structural Lie algebra \mathfrak{g} .

2. Let L be a Lie algebra bundle on smooth manifold M . The Lie algebroid $\mathcal{D}_{Der}(L)$ of covariant derivatives on $\Gamma^\infty(L)$ is a transitive Lie algebroid on M .
3. The Lie algebroid $\mathcal{D}(E)$ of covariant differential operators on the space of sections of vector bundle E .

As vector space is commutative Lie algebra, vector bundle E is also commutative Lie algebra bundle. Thus $\mathcal{D}(E)$ and $\mathcal{D}_{Der}(E)$ are identical in this case. In the following part of this article we use \mathfrak{g} to note Lie algebra and \mathfrak{h} to note commutative Lie algebra. All the Lie algebras we consider in this article are finite dimensional.

1.2 Functor $\mathcal{TLA}_{\mathfrak{g}}(\bullet)$

In [1], K. Mackenzie defines pullback of transitive Lie algebroid over smooth map $f : M' \rightarrow M$. It means that given a Lie algebra \mathfrak{g} there is the functor $\mathcal{TLA}_{\mathfrak{g}}(\bullet)$ such that with any manifold M it assigns the family $\mathcal{TLA}_{\mathfrak{g}}(M)$ of all transitive Lie algebroid with structural Lie algebra \mathfrak{g} .

Lemma 1.2.1. (See [1], page 248) *Let $0 \rightarrow L \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0$ be a transitive Lie algebroid on smooth manifold M . Then L is a Lie algebra bundle with respect to the braces structure on $\Gamma(L; M)$ induced from the braces on $\Gamma(A; M)$.*

Lemma 1.2.2. (See [1], page 100) *Let A be a transitive Lie algebroid on M and let $U \subset M$ be an open subset. Then the braces $\{, \} : \Gamma(A; M) \times \Gamma(A; M) \rightarrow \Gamma(A; M)$ restricted to $\Gamma(A_U; U) \times \Gamma(A_U; U) \rightarrow \Gamma(A_U; U)$ make A_U be a Lie algebroid on U called the restriction of A to U .*

Lemma 1.2.3. (See [1], page 317) *Consider a transitive Lie algebroid $0 \rightarrow L \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0$ on M with fixed structural Lie algebra \mathfrak{g} . Given any open covering $\{U_\alpha\}$ of M by contractible sets, for arbitrary α , there is an Lie algebroid isomorphism*

$$S_\alpha : TU_\alpha \bigoplus (U_\alpha \times \mathfrak{g}) \rightarrow A_{U_\alpha}$$

where $TU_\alpha \bigoplus (U_\alpha \times \mathfrak{g})$ is trivial Lie algebroid on U_α .

By using Lemma 1.2.1, Lemma 1.2.2, Lemma 1.2.3 and the method used in [2], we get the following theorem.

Theorem 1.2.4. *Let M and N be smooth manifolds. Given an arbitrary transitive Lie algebroid A on N . Let $f, g : M \rightarrow N$ are homotopic smooth maps. Then the pullback of A over f and g are Lie algebroid isomorphic, that is $f^! A \approx g^! A$.*

Hence the functor $\mathcal{TLA}_{\mathfrak{g}}(\bullet)$ is homotopy functor for fixed structural Lie algebra \mathfrak{g} . There exists a classifying space $\mathcal{B}_{\mathfrak{g}}$ such that $\mathcal{TLA}_{\mathfrak{g}}(M)$ has one to one correspondence with the family of homotopy classes of continuous maps $[M; \mathcal{B}_{\mathfrak{g}}]$. Here $\mathcal{B}_{\mathfrak{g}}$ is abstract and can be described in more or less understandable way (see [5]).

2 Obstruction

2.1 Cohomology

Definition 2.1.1. (see [1], page 107) Let A be an arbitrary Lie algebroid on a smooth manifold M and E is a vector bundle on M . Let $\mathcal{D}(E)$ be the Lie algebroid of covariant derivative on $\Gamma^\infty(E)$. A representation of A on E is a Lie algebroid homomorphism

$$\rho : A \rightarrow \mathcal{D}(E).$$

The cohomology space $\mathcal{H}^n(A, \rho, E), n \geq 0$ can be defined when the representation ρ is given (see [1], page 260). When A is TM , we denote the representation by $\nabla : TM \rightarrow E$. Then there is $\mathcal{H}^n(M, \nabla, E), n \geq 0$. The representation $\nabla : TM \rightarrow E$ can be regarded as a flat connection on E (see [1], page 109, page 186). Due to Lemma 1.1.6 and Lemma 1.2.2 in [3], the following theorem holds.

Theorem 2.1.2. Let E be a vector bundle on smooth manifold M and $\nabla : TM \rightarrow E$ be a representation of TM on E . Let $f : M' \rightarrow M$ be a smooth map between smooth manifold M' and M . Let $E' = f^*E$ be the pullback of vector bundle over f . Then

- (i) the representation ∇ induces a representation of TM' on E' noted by $\nabla' : TM' \rightarrow \mathcal{D}(E')$.
- (ii) the map f induces a homomorphism between cohomologies

$$f^* : \mathcal{H}^*(M, \nabla, E) \rightarrow \mathcal{H}^*(M', \nabla', E'),$$

where

$$\mathcal{H}^*(M, \nabla, E) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n(M, \nabla, E), \quad \mathcal{H}^*(M', \nabla', E') = \bigoplus_{n=0}^{\infty} \mathcal{H}^n(M', \nabla', E').$$

From fundamental differential geometry, the following theorem holds.

Theorem 2.1.3. Let E be a commutative Lie algebra bundle with fiber \mathfrak{h} . Let ∇ be a flat connection on it. Then ∇ induces the system of transition functions $\{\varphi_{\alpha\beta}\}$ for E that are locally constant. Then E can be seen as vector bundle with discrete structural group $\mathbf{Aut}(\mathfrak{h})_d$, and denoted by $E^\nabla \rightarrow M$. Here $\mathbf{Aut}(\mathfrak{h})_d$ is the group of all automorphisms of \mathfrak{h} , that is $\mathbf{Aut}(\mathfrak{h})$, with discrete topology.

2.2 Obstruction class

Let L be a Lie algebra bundle on smooth manifold M with fiber \mathfrak{g} . There is a commutative diagram (see [1]).

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & ZL & \xrightarrow{=} & ZL & & \\
 & & \downarrow i & & \downarrow i & & \\
 & & L & \xrightarrow{=} & L & & \\
 & & \downarrow ad & & \downarrow ad & & \\
 0 & \longrightarrow & \mathbf{Der}(L) & \xrightarrow{j} & \mathcal{D}_{Der}(L) & \xrightarrow{a} & TM \longrightarrow 0 \\
 & & \downarrow \mathfrak{h}^0 & & \downarrow \mathfrak{h} & & \downarrow = \\
 0 & \longrightarrow & \mathbf{Out}_{Der}(L) & \xrightarrow{\bar{j}} & \mathbf{Out}_{Der}(L) & \xrightarrow{\bar{a}} & TM \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

in which both rows and columns are exact.

Consider a coupling $\Xi : TM \rightarrow \mathbf{Out}_{Der}(L)$, that is the curvature tensor

$$R^\Xi : \Lambda^2(TM) \rightarrow \mathbf{Out}_{Der}(L)$$

defined by

$$R^\Xi(X, Y) = [\Xi(X), \Xi(Y)] - \Xi([X, Y])$$

for $X, Y \in \mathfrak{X}^1(M)$ is zero.

There is a lifting $\nabla_\Xi : TM \rightarrow \mathcal{D}_{Der}(L)$ of the coupling Ξ :

$$\begin{array}{ccc}
 L & & \\
 \downarrow ad & & \\
 \mathcal{D}_{Der}(L) & \xleftarrow{\nabla_\Xi} & TM \\
 \downarrow \mathfrak{h} & & \downarrow = \\
 \mathbf{Out}_{Der}(L) & \xleftarrow{\Xi} & TM \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

in which ∇ is vector bundle map.

Then for curvature tensor $R^{\nabla\Xi} : \Lambda^2(TM) \rightarrow \mathbf{Der}(L)$ defined by $R^{\nabla\Xi}(X, Y) = [\nabla_{\Xi}(X), \nabla_{\Xi}(Y)] - \nabla_{\Xi}([X, Y])$, the following diagram is commutative.

$$\begin{array}{ccc}
L & & \\
\downarrow ad & & \\
\mathbf{Der}(L) & \xleftarrow{R^{\nabla\Xi}} & \Lambda^2(TM) \\
\downarrow \mathfrak{h}^0 & \nearrow 0 & \\
\mathbf{Out}_{Der}(L) & & \\
\downarrow & & \\
0 & &
\end{array}$$

Since vertical column is exact there is a lifting of $R^{\nabla\Xi}$ that is a bundle map $\Omega : \Lambda^2(TM) \rightarrow L$ such that the diagram

$$\begin{array}{ccc}
L & & \\
\downarrow ad & \nearrow \Omega & \\
\mathbf{Der}(L) & \xleftarrow{R^{\nabla\Xi}} & \Lambda^2(TM) \\
\downarrow \mathfrak{h}^0 & \nearrow 0 & \\
\mathbf{Out}_{Der}(L) & & \\
\downarrow & & \\
0 & &
\end{array} \tag{1}$$

is commutative.

Define $d^{\nabla} : \Gamma(\Omega^n(M, L); M) \rightarrow \Gamma(\Omega^{n+1}(M, L); M)$ by

$$\begin{aligned}
d^{\nabla} f(X_1, X_2, \dots, X_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \nabla(X_i)(f(X_1, X_2, \dots, \hat{X}_i, \dots, X_{n+1})) \\
&\quad + \sum_{i < j} (-1)^{i+j} f([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{n+1})
\end{aligned}$$

here $f \in \Gamma(\Omega^n(M, L); M)$ and $X_1, X_2, \dots, X_{n+1} \in \mathfrak{X}^1$.

For Ω in diagram (1), $d^{\nabla\Xi}\Omega \in \Omega^3(M, ZL)$ and $d^{\nabla\Xi^{ZL}}(d^{\nabla}) = 0$ where ∇_{Ξ}^{ZL} is induced by ∇_{Ξ} (see [1]). Then define $Obs(\nabla_{\Xi}) = [d^{\nabla\Xi}(\Omega)] \in \mathcal{H}^3(M, \nabla_{\Xi}^{ZL}, ZL)$. The connection ∇_{Ξ}^{ZL} and cohomology class $Obs(\nabla_{\Xi})$ depend only on Ξ (see [1], page 273 and Theorem 7.2.12). Then the class $Obs(\nabla_{\Xi})$ is called the *obstruction class* of the coupling Ξ , and is denoted by $Obs(\Xi)$.

Theorem 2.2.1. (The functorial property) *Let L be a finite dimensional Lie algebra bundle on a smooth manifold M . Let M' be a smooth manifold and $f : M' \rightarrow M$ is smooth map. Let $L' = f^*L$ be the pullback of Lie algebra bundle*

over f . Consider a coupling $\Xi : TM \rightarrow \mathbf{Out}\mathcal{D}_{Der}L$. Then Ξ induces a coupling $\Xi' : TM' \rightarrow \mathbf{Out}\mathcal{D}_{Der}L'$ and f induces a homomorphism

$$f^* : \mathcal{H}^*(M, \Xi, ZL) \rightarrow \mathcal{H}^*(M', \Xi', ZL').$$

Further more the obstruction class $Obs(\Xi') \in \mathcal{H}^3(M', \Xi', ZL')$ satisfies the condition

$$f^*(Obs(\Xi)) = Obs(\Xi')$$

Definition 2.2.2. An extension of TM by Lie algebra bundle L is an exact sequence of Lie algebroid over M

$$0 \rightarrow L \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0.$$

Theorem 2.2.3. (see [1], corollary 7.3.9) Let L be a Lie algebra bundle on M . Let $\Xi : TM \rightarrow \mathbf{Out}\mathcal{D}_{Der}(L)$ be a coupling. Then, if $Obs(\Xi) = 0$, there is a Lie algebroid extension

$$0 \rightarrow L \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0$$

of TM by L inducing the coupling Ξ .

Corollary 2.2.4. Let E be a vector bundle over M (that is the Lie algebra bundle with commutative Lie algebra). There is a Lie algebroid extension

$$0 \rightarrow E \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0$$

if and only if the bundle E is flat.

Proof. Suppose that the extension exists

$$0 \rightarrow E \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0$$

Let $\lambda : TM \rightarrow A$ be a splitting. Define

$$\nabla^\lambda : \mathfrak{X}^1(M) \times \Gamma^\infty(E; M) \rightarrow \Gamma^\infty(E; M)$$

by the formula

$$\nabla_X^\lambda(\mu) = \{\lambda(X), \mu\}.$$

Then

$$\begin{aligned} R^{\nabla^\lambda}(X, Y)(\mu) &= [\nabla_X^\lambda, \nabla_Y^\lambda](\mu) - \nabla_{[X, Y]}^\lambda(\mu) = \\ &= \{[\lambda(X), \lambda(Y)] - \lambda([X, Y]), \mu\} = 0 \end{aligned}$$

for arbitrary $X, Y \in \mathfrak{X}^1(M)$, $\mu \in \Gamma(E; M)$ since $a([\lambda(X), \lambda(Y)] - \lambda([X, Y])) = 0$ that is $[\lambda(X), \lambda(Y)] - \lambda([X, Y]) \in \Gamma(E; M)$ and the structural Lie algebra is commutative.

Conversely. If E is flat, there is a flat connection ∇ on E which also is a representation of the Lie algebroids

$$\nabla : TM \rightarrow \mathcal{D}(E),$$

that is $R^\nabla(X, Y) = 0$.

By definition of obstruction class this means that $Obs(\nabla) = 0 \in \mathcal{H}^3(M, \nabla, E)$. Then there exist Lie algebroid extensions. \square

3 Characteristic Classes

In this section a system of characteristic classes of transitive Lie algebroid with commutative adjoint bundle will be described. Then they will be compared with characteristic classes derived from Chern-Weil homomorphism by J.Kubarski ([3]). As a matter of fact we show that the Chern-Weil homomorphism does not cover all characteristic classes from categorical point of view.

3.1 A system of characteristic classes for commutative case

Let \mathfrak{h} be a finite dimensional commutative Lie algebra. Let $\mathbf{Aut}(\mathfrak{h})_d$ be the group $\mathbf{Aut}(\mathfrak{h})$ with discrete topology. The functor $Vector_d^{\mathfrak{h}}(\bullet)$ associates with each paracompact topology space X the set $Vector_d^{\mathfrak{h}}(X)$ of all vector bundle with structural group $\mathbf{Aut}(\mathfrak{h})_d$. Let $E^\infty \rightarrow B\mathbf{Aut}(\mathfrak{h})_d$ be universal bundle with group $\mathbf{Aut}(\mathfrak{h})_d$ and let $B\mathbf{Aut}(\mathfrak{h})_d$ be the classifying space.

Lemma 3.1.1. (See [6], Definition 11.1, Theorem 11.2, Theorem 12.2) *There is a bijection between $Vector_d^{\mathfrak{h}}(X)$ and the homotopy classes of continuous maps $[X; B\mathbf{Aut}(\mathfrak{h})_d]$.*

Let M be a smooth manifold and

$$0 \rightarrow E \xrightarrow{j} A \xrightarrow{a} TM \rightarrow 0 \quad (2)$$

be a transitive Lie algebroid with fixed structural commutative Lie algebra $\mathfrak{h} = R^n$. Let $\lambda : TM \rightarrow A$ be a splitting. Define $\nabla = \nabla^\lambda$ by a formula $\nabla_X^\lambda(\mu) = \{\lambda(X), \mu\}$. The bundle E possesses a flat structure $E^\nabla \in Vector_d^{\mathfrak{h}}(M)$. Let $f : M' \rightarrow M$ be a smooth map and $f^!A$ be the pullback of Lie algebroid A over f , that is

$$0 \rightarrow f^*E \xrightarrow{j'} f^!A \xrightarrow{a'} TM' \rightarrow 0. \quad (3)$$

Let $\lambda' : TM' \rightarrow f^!A$ be a splitting. Define $\nabla' = \nabla^{\lambda'}$ on f^*E and f^*E is corresponding to $(f^*E)^{\nabla'}$.

Lemma 3.1.2. (i) ∇ and ∇' are independent of the choice of λ and λ' ,

(ii) *The bundle $(f^*E)^{\nabla'}$ is the pullback of E^∇ over $f : M' \rightarrow M$ in the category of vector bundle with discrete structural group $\mathbf{Aut}(\mathfrak{h})_d$.*

Proof. Statement (i) is obvious.

(ii) : Consider the splitting of transitive Lie algebroid (3)

$$\lambda' : TM' \rightarrow f^!A$$

by the formula

$$\lambda'(X') = (X', \lambda(Tf(X'))),$$

$$X' \in TM'.$$

Let $\sum_i h_i \cdot (\mu_i \circ f) \in \Gamma(f^*E; M)$, here $h_i \in C^\infty(M')$, $\mu_i \in \Gamma(E; M)$. Then

$$\nabla_{X'}^{\lambda'}(\sum_i h_i \cdot (\mu_i \circ f)) = \sum_i X'(h_i) \cdot (\mu_i \circ f) + \sum_i h_i \cdot (\nabla_{Tf(X')}^\lambda(\mu_i) \circ f) \quad (4)$$

As ∇^λ is flat connection, there exist chart $\{\varphi_\alpha : E_{U_\alpha} \rightarrow U_\alpha \times \mathfrak{h}\}_{\alpha \in \Delta}$ which satisfies the condition

$$\varphi_\alpha(\nabla_X^\lambda(\mu_\alpha)) = X(\varphi_\alpha(\mu_\alpha)) \quad (5)$$

for arbitrary $\mu_\alpha \in \Gamma(E_{U_\alpha}; U_\alpha)$, $X \in \mathfrak{X}^1(M)$.

Consider $\mu \in \Gamma(E_{U_\alpha \cap U_\beta}; U_\alpha \cap U_\beta)$. Then

$$X(\varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha(\mu)) = X(\varphi_\beta(\mu)) = \varphi_\beta(\nabla_X^\lambda(\mu)) = \varphi_\beta \circ \varphi_\alpha^{-1}(\varphi_\alpha(\nabla_X^\lambda(\mu))).$$

Then

$$X(\varphi_\beta \circ \varphi_\alpha^{-1} \circ \varphi_\alpha(\mu)) = \varphi_\beta \circ \varphi_\alpha^{-1}(X(\varphi_\alpha(\mu))) \quad (6)$$

Thus the transition functions $\{\varphi_{\alpha\beta}\}_{\alpha, \beta \in \Delta}$ are all locally constant.

Let $\{V'_\alpha = f^{-1}(U_\alpha)\}_{\alpha \in \Delta}$ be atlas of charts on M' . Define the homomorphism of $C^\infty(V'_\alpha)$ -modules

$$\psi_\alpha : \Gamma(f^*E|_{V'_\alpha}; V'_\alpha) \rightarrow \Gamma(V'_\alpha \times \mathfrak{h}; V'_\alpha)$$

defined by the formula

$$\psi_\alpha(h_{\alpha,i} \cdot (\mu_\alpha^i \circ f)) = h_{\alpha,i} \cdot \varphi_\alpha(\mu_\alpha^i) \circ f$$

for $h_{\alpha,i} \cdot (\mu_\alpha^i \circ f) \in \Gamma(f^*E|_{V'_\alpha}; V'_\alpha)$, where $h_{\alpha,i} \in C^\infty(V'_\alpha)$, $\mu_\alpha^i \in \Gamma(E_{U_\alpha}; U_\alpha)$.

As φ_α is vector bundle isomorphism, ψ_α induces a vector bundle isomorphism. Then $\{V'_\alpha, \psi_\alpha : f^*E|_{V'_\alpha} \rightarrow V'_\alpha \times \mathfrak{h}\}_{\alpha \in \Delta}$ is a chart for f^*E . Consider a vector field $X' \in \mathfrak{X}^1(M')$. Then

$$\begin{aligned} & \psi_\alpha(\nabla_{X'}^{\lambda'}(h_{\alpha,i} \cdot (\mu_\alpha^i \circ f))) \\ &= \psi_\alpha(X'(h_{\alpha,i}) \cdot (\mu_\alpha^i \circ f) + h_{\alpha,i} \cdot (\nabla_{Tf(X')}^\lambda(\mu_\alpha^i) \circ f)) = \\ &= X'(h_{\alpha,i}) \cdot (\varphi_\alpha(\mu_\alpha^i) \circ f) + h_{\alpha,i} \cdot (Tf(X')(\varphi_\alpha(\mu_\alpha^i)) \circ f) \\ &= X'(h_{\alpha,i} \cdot (\varphi_\alpha(\mu_\alpha^i) \circ f)) = X'(\psi_\alpha(h_{\alpha,i} \cdot (\mu_\alpha^i \circ f))) \end{aligned}$$

The transition functions

$$\psi_{\alpha\beta} : V'_\alpha \cap V'_\beta \rightarrow \mathbf{Aut}(\mathfrak{h})_d$$

are defined by

$$\psi_{\alpha\beta}(x') = \varphi_{\alpha\beta}(f(x'))$$

for $x' \in V'_\alpha \cap V'_\beta$.

So $(f^*E)^{\nabla'}$ is the pullback of E^∇ over $f : M' \rightarrow M$ in the category of vector bundle with discrete structural group $\mathbf{Aut}(\mathfrak{h})_d$. \square

The Lemma 3.1.2 shows that the following definition is corrected.

Definition 3.1.3. Let \mathfrak{h} be a commutative Lie algebra and M be a smooth manifold. Let $A \in \mathcal{TLA}_{\mathfrak{h}}(M)$, with splitting λ . Let E^{∇^λ} be the correspondent Lie algebra bundle with flat structure. Let $\theta : \text{Vector}_{\mathfrak{h}}^d(M) \rightarrow [M; B\mathbf{Aut}_{(\mathfrak{h})_d}]$ be the bijection defined in Lemma 3.1.1. Then $\theta(E^{\nabla^\lambda}) = [f] \in [M; B\mathbf{Aut}_{(\mathfrak{h})_d}]$ induces a homomorphism

$$f^* : H^*(B\mathbf{Aut}_{(\mathfrak{h})_d}; R) \rightarrow H^*(M; R).$$

The class $f^*(c) \in H^*(M; R)$ is characteristic class of A , for arbitrary $c \in H^*(B\mathbf{Aut}_{(\mathfrak{h})_d}; R)$.

3.2 Chern-Weil homomorphism

Definition 3.2.1. (see [3], page 17) Given a transitive Lie algebroid $(A, q, M, \{\cdot, \cdot\}, a)$ with adjoint bundle L . The adjoint representation of a transitive Lie algebroid A is

$$ad : A \rightarrow \mathcal{D}(L)$$

defined by

$$ad(\xi)(\nu) = \{\xi, \nu\}$$

for $\xi \in \Gamma(A; M)$, $\nu \in \Gamma(L; M)$. Let L^* be dual bundle of L and $\bigvee^k L^*$ is k -th symmetric power of L^* (see [7], page 191). The adjoint representation ad can rise to

$$\bigvee^k ad^{\natural} : A \rightarrow \mathcal{D}(\bigvee^k L^*)$$

such that

$$\begin{aligned} & \langle \bigvee^k ad^{\natural}(\xi)(\varphi), \nu^1 \vee \nu^2 \vee \dots \vee \nu^k \rangle = \\ & = a(\xi)(\langle \varphi, \nu^1 \vee \nu^2 \vee \dots \vee \nu^k \rangle) - \sum_{i=1}^k \langle \varphi, \nu^1 \vee \dots \vee \{\xi, \nu^i\} \vee \dots \vee \nu^k \rangle \end{aligned}$$

for $\xi \in \Gamma(A; M)$, $\varphi \in \Gamma(\bigvee^k L^*; M)$, $\nu^i \in \Gamma(L; M)$.

Remark 3.2.2. Here we only consider the vector bundle structure of L that is commutative Lie algebra structure. Hence we use notation $\mathcal{D}(L)$ and $\mathcal{D}(\bigvee^k L^*)$.

Definition 3.2.3. (see [3], Definition 2.3.1) Given an arbitrary transitive Lie algebroid $0 \rightarrow L \rightarrow A \xrightarrow{a} TM \rightarrow 0$. Let L^* be dual bundle of L . A section $\varphi \in \Gamma(\bigvee^k L^*; M)$ is called $\bigvee^k ad^{\natural}$ -invariant if $\bigvee^k ad^{\natural}(\xi)(\varphi) = 0$ for all $\xi \in \Gamma(A; M)$. The space of all $\bigvee^k ad^{\natural}$ -invariant sections of $\bigvee^k L^*$ is denoted by $\Gamma^I(\bigvee^k L^*; M)$.

Definition 3.2.4 (Chern-Weil homomorphism). (see [3], page 29) Given a transitive Lie algebroid $(A, q, M, \{\cdot, \cdot\}, a)$ with adjoint bundle L . Let $\lambda : TM \rightarrow$

A be a splitting and $R^\lambda \in \Omega^2(M; L)$ be the curvature tensor, $R^\lambda(X, Y) = \{\lambda(X), \lambda(Y)\} - \lambda([X, Y])$.

Define a homomorphism of $C^\infty(M)$ -modules

$$\chi_{(A, \lambda), I} : \Gamma^I(\bigvee^k L^*; M) \rightarrow \Omega^{2k}(M)$$

by the formula

$$\chi_{(A, \lambda), I} = \frac{1}{k!} \langle \varphi, R_\lambda \vee R_\lambda \vee \dots \vee R_\lambda \rangle$$

for $\varphi \in \Gamma(\bigvee^k L^*; M)$. Here

$$\begin{aligned} \langle \varphi, R_\lambda \vee \dots \vee R_\lambda \rangle (X_1, X_2, \dots, X_{2k}) = \\ = \langle \varphi, \frac{1}{2^k} \sum_{\sigma} (-1)^\sigma R_\lambda(X_{\sigma(1)}, X_{\sigma(2)}) \vee R_\lambda(X_{\sigma(3)}, X_{\sigma(4)}) \vee \dots \vee R_\lambda(X_{\sigma(2k-1)}, X_{\sigma(2k)}) \rangle \end{aligned}$$

The forms from the image of $\chi_{(A, \lambda), I}$ is closed (see [3], proposition 4.1.2). Then Chern-Weil homomorphism is defined by the composition

$$h_{(A, \lambda)} : \bigoplus_{k \geq 0} \Gamma^I(\bigvee^k L^*; M) \xrightarrow{\chi_{(A, \lambda), I}} \mathbf{Ker} d^{\nabla^\lambda} \xrightarrow{i} H_{DRam}^*(M; R).$$

The Chern-Weil homomorphism has functorial property and is independent of the choice of splitting (see [3], theorem 4.2.2, theorem 4.3.7). Then $h_{(A, \lambda)}$ can be denoted as

$$h_A : \bigoplus_{k \geq 0} \Gamma^I(\bigvee^k L^*; M) \rightarrow H_{DRam}^*(M; R).$$

3.3 Example

The following example shows that the Chern-Weil homomorphism does not cover all categorical characteristic classes.

Consider a flat 1-dimensional vector bundle E over a torus $T^2 = S^1 \times S^1$. We will consider E as a Lie algebra bundle with commutative Lie algebra $\mathfrak{h} \approx \mathbf{R}^1$. The structural group of the bundle E is the group $R^* = R \setminus \{0\}$ with discrete topology. The flat structure on E is defined by an atlas of charts $\{U_\alpha\}$ with trivialization of the bundle E on each chart U_α such that all transition function are locally constant. Transition functions are fully defined by a representation of the fundamental $\pi_1(T^2)$ in the structural group $\mathbf{Aut}(\mathfrak{h})_d$, $\rho : \pi_1(T^2) \rightarrow \mathbf{Aut}(\mathfrak{h})_d$.

There is a flat connection ∇ on $E \rightarrow T^2$ which corresponds to the flat structure on E . This means that the connection on each chart U_α (after trivialization of the bundle E) coincides with usual derivative ($\nabla_X = \frac{\partial}{\partial X}$).

Construct a Lie algebroid \mathcal{A} :

$$0 \rightarrow E \rightarrow T(T^2) \bigoplus E \rightarrow T(T^2) \rightarrow 0$$

with bracket

$$\{(X, \mu), (Y, \nu)\} = ([X, Y], \nabla_X(\nu) - \nabla_Y(\mu) + \Omega(X, Y))$$

for $(X, \mu), (Y, \nu) \in \Gamma(T(T^2) \oplus E; T^2)$. Here $\Omega \in \mathbf{Ker} \, d^\nabla \subset \Omega^2(T^2, E)$. Let E^* be the bundle dual to E . Let $f \in \Gamma^I(E^*; T^2)$. Then

$$\begin{aligned} ad^\sharp((X, \nu))(f)(\mu) &= X(f(\mu)) - f(\{X \oplus \nu, 0 \oplus \mu\}) = \\ &= X(f(\mu)) - f(\nabla_X(\mu)) = 0 \end{aligned} \quad (7)$$

for arbitrary $\mu \in \Gamma(E; T^2)$, $(X, \nu) \in \Gamma(T(T^2) \oplus E; T^2)$. Hence locally on the chart U_α the function f is constant.

This means that in the case of nontrivial representation the space $\Gamma^I(E^*; T^2)$ has only trivial element. Thus the characteristic class for \mathcal{A} defined by Chern-Weil homomorphism by J.Kubarski is trivial.

On the other hand the characteristic classes due to definition 3.1.3 are not trivial. Namely the structural group $\mathbf{Aut}(\mathfrak{h})_d$ is isomorphic to $R \setminus \{0\} \approx \mathbb{Z}_2 \times R$.

Hence the classifying space for vector bundle with discrete structural group \mathfrak{h} is $B_{\mathbb{Z}_2} \times B_R$. We have $B_{\mathbb{Z}_2} \sim \mathbb{RP}^\infty$. The group R is a direct sum $R \approx \bigoplus_{\alpha \in A} \mathbb{Q}_\alpha$ where each group \mathbb{Q}_α is isomorphic to rational numbers, $\mathbb{Q}_\alpha \approx \mathbb{Q}$. The group \mathbb{Q} is isomorphic to the direct limits $\mathbb{Q} = \varinjlim (\mathbb{Z}_n, \omega_n)$, where all \mathbb{Z}_n are isomorphic to \mathbb{Z} , and $\omega_n : \mathbb{Z}_n \rightarrow \mathbb{Z}_{n+1}, \omega_n(k) = (n+1)k$.

Thus the classifying space B_R can be represent as a direct limits

$$B_R = \varinjlim_{b \subset B} \mathbb{T}_b,$$

where each $b \in B$ is a finite collection of indexes

$$b = \{\alpha_1, n_1, \alpha_2, n_2, \dots, \alpha_k, n_k\}, \alpha_j \in A, n_j \in \mathbb{Z}$$

that are ordered in the natural way, $\mathbb{T}_b = \prod_{j=1}^k S_{\alpha_j, n_j}^1 \approx \mathbb{T}^k$.

The cohomology group $H^*(B_{\mathbf{Aut}(\mathfrak{h})_d}; R)$ can be describe in the following way:

$$\begin{aligned} H^*(B_{\mathbb{Z}_2}; R) &\approx R; \\ H^*(B_R; R) &\approx \varprojlim_{b \subset B} H^*(\mathbb{T}_b; R). \end{aligned}$$

The representation $\rho : \pi_1(T^2) \rightarrow \mathbf{Aut}(\mathfrak{h})_d$ induces the map

$$B_\rho : \mathbb{T}_2 \rightarrow B_{\mathbb{Z}_2} \times B_R,$$

and the homomorphism in cohomology

$$B_\rho^* : H^*(B_{\mathbb{Z}_2} \times B_R; R) \rightarrow H^*(\mathbb{T}_2; R).$$

Lemma 3.3.1 (Key lemma). *The homomorphism B_ρ^* is surjective.*

The example show that Chern-Weil homomorphism cannot define all characteristic classes for transitive Lie algebroid.

Remark 3.3.2. *This example show that there is a natural problem to generalize the Chern-Weil homomorphism for non trivial flat bundle ZL of local coefficients for cohomologies that contain characteristic classes.*

This work is partly supported by scientific program for the Chief International Academic Adviser of the Harbin institute of technology (2011-2014)(China) and Russian foundation of Basic research grant No.11-01-00057-a.

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